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Mirage torsion

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Abstract: $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models admit the introduction of a discrete torsion phase. We find that models with discrete torsion have an alternative description in terms of torsionless models. More specifically, discrete torsion can be 'gauged away' by changing the shifts by lattice vectors. Similarly, a large class of the so-called generalized discrete torsion phases can be traded for changing the background fields (Wilson lines) by lattice vectors. We further observe that certain models with generalized discrete torsion are equivalent to torsionless models with the same gauge embedding but based on different compactification lattices. We also present a method of classifying heterotic $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds.

Keywords: Superstrings and Heterotic Strings, Compactification and String Models.

Contents

1. Introduction

Among the currently available frameworks, superstring theory appears to have the greatest prospects for yielding a unified description of nature. Optimistically one may hope to identify a string compactification that reproduces all observations. The perhaps simplest way to obtain a chiral spectrum in four dimensions, as required by observation, is to compactify on an orbifold [1, 2]. Although it is straightforward to compute orbifold spectra, a deep understanding of these constructions, including an interpretation of the zero-modes, is harder to obtain. Obstructions arise from the large number of possible gauge embeddings and geometries, as well as other degrees of freedom. The classification of gauge embeddings has been accomplished only in prime orbifolds. The generalization to $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with or without Wilson lines has not been discussed in the literature so far. $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds are particularly rich as they can be generalized by turning on certain phases which are known as discrete torsion [3–7].

Aiming at a systematic understanding of heterotic $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, we set out to survey the possibilities arising in these constructions. In the course of our investigations we obtain rather surprising results. First of all, discrete torsion can be 'gauged away' in the sense that models with discrete torsion have an alternative description in terms of torsionless models. Moreover, we shall see that the so-called 'non-factorizable' orbifolds are equivalent to factorizable orbifolds with (generalized) discrete torsion or different gauge embedding.

This paper is organized as follows. In section 2 we collect some basic facts on the construction of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models. We encourage readers who are familiar with the construction of orbifolds to skip this section. In section 3, we establish the equivalence between switching on a discrete torsion phase and changing the gauge embedding by elements of the weight lattice. Section 4 is devoted to the generalization to orbifolds with Wilson lines. In section 5 we outline a prescription for a classification of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold models. Finally, section 6 contains a discussion of our results. Some issues concerning the transformation phases are discussed in the appendix.

2. $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold compactifications

2.1 Setup

Let us start by reviewing some basic facts on orbifold compactifications [2]. To construct an orbifold, one first considers a d-dimensional torus $\mathbb{T}^d$, which can be understood as $\mathbb{R}^d/\Gamma$, i.e. as the d-dimensional space with points differing by lattice vectors $e_\alpha \in \Gamma$ identified. In this study we will take $d = 6$ in order to arrive at an effective four-dimensional theory at low energies. If the torus lattice enjoys one or more discrete rotational symmetries comprising the point group P, one can define an orbifold as the quotient $\mathbb{O} = \mathbb{T}^6/P$. Equivalently one can describe the orbifold by

$$\mathbb{O} \; = \; \frac{\mathbb{R}^6}{S} \; , \tag{2.1}$$

where S is the space group. Space group elements consist of discrete rotations and translations by lattice vectors e_α. We will be mostly interested in $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, in which the torus lattice has two discrete rotational symmetries described by the independent twists θ and ω, whereby $\theta^N = \omega^M = \mathbb{1}$ and N is a multiple of M. Space group elements $g \in S$ are then given by $g = (\theta^{k_1}\,\omega^{k_2}, n_\alpha\, e_\alpha)$ where $0 \leq k_1 \leq N-1$, $0 \leq k_2 \leq M-1$ and $n_\alpha \in \mathbb{Z}$. Further, we restrict our analysis to models with SU(3) holonomy, where the rotations can be diagonalized,

$$\theta\, z^i \;=\; \exp(2\pi\mathrm{i}\, v_1^i)\, z^i \quad \text{and} \quad \omega\, z^i \;=\; \exp(2\pi\mathrm{i}\, v_2^i)\, z^i \; , \tag{2.2}$$

with $z^{1,2,3}$ being the complex coordinates of the compact space, and $\sum_i v_{1,2}^i = 0$. Unless stated otherwise, we use

$$v_1 \;=\; \frac{1}{N}(1,0,-1;0) \quad \text{and} \quad v_2 \;=\; \frac{1}{M}(0,1,-1;0)\; . \tag{2.3}$$

The space group action is to be embedded in the gauge degrees of freedom according to

$$g = (\theta^{k_1}\,\omega^{k_2}, n_\alpha\, e_\alpha) \;\hookrightarrow\; (\mathbb{1}, V_g) \qquad V_g = k_1\, V_1 + k_2\, V_2 + n_\alpha\, A_\alpha \; , \tag{2.4}$$

where V_1, V_2 are the shifts, A_α are the Wilson lines, and V_g denotes the local shift corresponding to the twist $v_g = k_1 v_1 + k_2 v_2$. Due to the embedding, they have to be of appropriate orders:

$$N\,V_1 \;\in\; \Lambda\,, \quad M\,V_2 \;\in\; \Lambda\,, \quad N_\alpha\,A_\alpha \;\in\; \Lambda\,. \tag{2.5}$$

Here Λ is the $\mathrm{E}_8 \times \mathrm{E}_8$ or $\mathrm{Spin}(32)/\mathbb{Z}_2$ weight lattice,[1] and N_α denotes the order of the Wilson line A_α, which is constrained by geometry.

Modular invariance of one-loop amplitudes imposes strong conditions on the shifts and Wilson lines. In $\mathbb{Z}_N$ orbifolds, the shift V and the twist v must fulfill [2, 3]:

$$N\,\left(V^2 - v^2\right) \;=\; 0 \quad \mathrm{mod}\ 2\,. \tag{2.6}$$

In $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with Wilson lines, modular invariance, together with consistency requirements (see appendix A), requires

$$N\,\left(V_1^2 - v_1^2\right) = 0 \quad \mathrm{mod}\ 2\,, \tag{2.7a}$$
$$M\,\left(V_2^2 - v_2^2\right) = 0 \quad \mathrm{mod}\ 2\,, \tag{2.7b}$$
$$M\,\left(V_1 \cdot V_2 - v_1 \cdot v_2\right) = 0 \quad \mathrm{mod}\ 2\,, \tag{2.7c}$$
$$N_\alpha\,\left(A_\alpha \cdot V_i\right) = 0 \quad \mathrm{mod}\ 2\,, \tag{2.7d}$$
$$N_\alpha\,\left(A_\alpha^2\right) = 0 \quad \mathrm{mod}\ 2\,, \tag{2.7e}$$
$$Q_{\alpha\beta}\,\left(A_\alpha \cdot A_\beta\right) = 0 \quad \mathrm{mod}\ 2 \quad (\alpha \neq \beta)\,, \tag{2.7f}$$

where $Q_{\alpha\beta} \equiv \gcd(N_\alpha, N_\beta)$ denotes the greatest common divisor of N_α and N_β.

2.2 Spectrum

Given a compactification lattice, the discrete rotations described by $v_{1,2}$, shifts and Wilson lines, there exists a standard procedure to calculate the massless spectrum (cf. [8–13]). The Hilbert space decomposes in untwisted and various twisted sectors, denoted by U and $T_{(k_1,k_2,n_\alpha)}$, respectively. The gauge group after compactification is generated by the 16 Cartan generators plus roots $p \in \Lambda$ ($p^2 = 2$) fulfilling

$$p \cdot V_i \;=\; 0 \quad \mathrm{mod}\ 1 \quad \forall i\,, \quad p \cdot A_\alpha \;=\; 0 \quad \mathrm{mod}\ 1 \quad \forall \alpha\,. \tag{2.8}$$

Chiral untwisted sector states are described by $p \in \Lambda$ and $q \in \Lambda_{\mathrm{SO(8)}}$ ($q^2 = 1$) satisfying

$$p \cdot V_i - q \cdot v_i = 0 \quad \mathrm{mod}\ 1\,, \quad i \in 1, 2\,, \tag{2.9a}$$
$$q \cdot v_i \neq 0\,, \quad i = 1 \text{ and/or } 2\,, \tag{2.9b}$$
$$p \cdot A_\alpha = 0 \quad \mathrm{mod}\ 1 \quad \forall \alpha\,. \tag{2.9c}$$

Twisted sector zero modes are associated to the inequivalent 'constructing elements' $g = (\theta^{k_1}\omega^{k_2}, n_\alpha e_\alpha) \in S$, corresponding to the inequivalent fixed points and fixed planes. For

[1] Since these lattices are self-dual, we denote the root and weight lattice by the same symbol.

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each such g one solves the mass equations

$$\frac{1}{8}m_{\mathrm{L}}^2 = \frac{1}{2}p_{\mathrm{sh}}^2 - 1 + \omega_i\,\widetilde{N}_{g,i} + \overline{\omega}_i\,\widetilde{N}_{g,i}^* + \delta c \overset{!}{=} 0\,, \tag{2.10a}$$

$$\frac{1}{8}m_{\mathrm{R}}^2 = \frac{1}{2}q_{\mathrm{sh}}^2 - \frac{1}{2} + \delta c \overset{!}{=} 0\,, \tag{2.10b}$$

with the shifted momenta ($p \in \Lambda, q \in \Lambda_{\mathrm{SO}(8)}$)

$$p_{\mathrm{sh}} = p + V_g\,, \tag{2.11a}$$

$$q_{\mathrm{sh}} = q + v_g\,. \tag{2.11b}$$

Here $\omega_i = (v_g)_i \mod 1$ and $\overline{\omega}_i = -(v_g)_i \mod 1$, such that $0 < \omega_i, \overline{\omega}_i \leq 1$. Moreover, $\widetilde{N}_{g,i}$ and $\widetilde{N}_{g,i}^*$ are integer oscillator numbers. Finally, $\delta c = \frac{1}{2}\sum_i \omega_i\,(1-\omega_i)$.

The states $|q_{\mathrm{sh}}\rangle_{\mathrm{R}} \otimes |p_{\mathrm{sh}}\rangle_{\mathrm{L}}$, where q_{sh} and p_{sh} are solutions of the mass equations (2.10), are subject to certain invariance conditions: commuting elements $h = (\theta^{t_1}\omega^{t_2}, m_\alpha e_\alpha)$, with $[g,h] = 0$, have to act as the identity on physical states. This leads to the projection condition

$$|q_{\mathrm{sh}}\rangle_{\mathrm{R}} \otimes |p_{\mathrm{sh}}\rangle_{\mathrm{L}} \overset{h}{\longmapsto} \Phi\,|q_{\mathrm{sh}}\rangle_{\mathrm{R}} \otimes |p_{\mathrm{sh}}\rangle_{\mathrm{L}} \overset{!}{=} |q_{\mathrm{sh}}\rangle_{\mathrm{R}} \otimes |p_{\mathrm{sh}}\rangle_{\mathrm{L}}\,. \tag{2.12}$$

Here the transformation phase Φ is given by

$$\Phi \equiv e^{2\pi\mathrm{i}\,[p_{\mathrm{sh}}\cdot V_h - q_{\mathrm{sh}}\cdot v_h + (\widetilde{N}_g - \widetilde{N}_g^*)\cdot v_h]}\,\Phi_{\mathrm{vac}}\,, \tag{2.13}$$

where (cf. appendix A)

$$\Phi_{\mathrm{vac}} = e^{2\pi\mathrm{i}\,[-\frac{1}{2}(V_g\cdot V_h - v_g\cdot v_h)]}\,. \tag{2.14}$$

Equation (2.12) states that the transformation phase Φ has to vanish, which will be important for the following discussion.

3. Brother models and discrete torsion

In this section we start by examining a new possibility to find inequivalent models. We discuss under what circumstances models with shifts differing by lattice vectors have different spectra and are thus inequivalent. Then we review the concept of *discrete torsion*, and clarify its relation to models in which shifts differ by lattice vectors.

3.1 Brother models

Let us start by clarifying under which conditions two models M and M′ are equivalent. First, we restrict to the case without Wilson lines, where the models M and M′ are described by the set of shifts (V_1, V_2) and (V_1', V_2'), respectively. Clearly, if the shifts are related by Weyl reflections, i.e.

$$(V_1', V_2') = (W\,V_1, W\,V_2)\,, \tag{3.1}$$

where W represents a series of Weyl reflections, one does obtain equivalent models. Let us now turn to comparing the spectra of two models M and M′, where

$$(V_1', V_2') = (V_1 + \Delta V_1, V_2 + \Delta V_2)\,, \tag{3.2}$$

with $\Delta V_1, \Delta V_2 \in \Lambda$. For future reference, we call models related by equation (3.2) '*brother models*'.

Brother models are also subject to modular invariance constraints. For the sake of keeping the expressions simple, we restrict here to models fulfilling the following (stronger) conditions:

$$V_i^2 - v_i^2 = 0 \mod 2 \quad (i = 1, 2) \,, \tag{3.3a}$$

$$V_1 \cdot V_2 - v_1 \cdot v_2 = 0 \mod 2 \,. \tag{3.3b}$$

(In section 4 we will relax these conditions.) Equations (3.3) imply that $\Phi_{\text{vac}} = 1$ in the transformation phase (2.13). The condition that (V_1', V_2') fulfill (3.3) leads to the following constraints on $(\Delta V_1, \Delta V_2)$:

$$V_i \cdot \Delta V_i = 0 \mod 1 \qquad i = 1, 2 \,, \tag{3.4a}$$

$$V_1 \cdot \Delta V_2 + \Delta V_1 \cdot V_2 + \Delta V_1 \cdot \Delta V_2 = 0 \mod 2 \,. \tag{3.4b}$$

Consider now the massless spectrum corresponding to the constructing element

$$g \;=\; (\theta^{k_1}\omega^{k_2}, n_\alpha e_\alpha) \;\in\; S \tag{3.5}$$

of the models M and M$'$. For simplicity, we restrict our attention to non-oscillator states. Physical states arise from tensoring together left- and right-moving solutions of the masslessness condition equation (2.10),

$$|q + k_1 v_1 + k_2 v_2\rangle_{\text{R}} \otimes |p + k_1 V_1 + k_2 V_2\rangle_{\text{L}} \quad \text{for M} \,, \tag{3.6}$$

$$|q + k_1 v_1 + k_2 v_2\rangle_{\text{R}} \otimes |p' + k_1 V_1' + k_2 V_2'\rangle_{\text{L}} \quad \text{for M}' \,, \tag{3.7}$$

where $p' = p - k_1\Delta V_1 - k_2 \Delta V_2$ and the shifted momenta of the left-movers are identical for M and M$'$. According to equation (2.13) with $\Phi_{\text{vac}} = 1$, these massless states transform under the action of a commuting element

$$h \;=\; (\theta^{t_1}\omega^{t_2}, m_\alpha e_\alpha) \in S \quad \text{with} \quad [h, g] \;=\; 0 \tag{3.8}$$

with the phases

$$\begin{aligned} \Phi &= e^{2\pi \mathrm{i}\,[(p+k_1V_1+k_2V_2)\cdot(t_1V_1+t_2V_2)-(q+k_1v_1+k_2v_2)\cdot(t_1v_1+t_2v_2)]} && \text{for M} \,, \\ \Phi' &= e^{2\pi \mathrm{i}\,\left[(p'+k_1V_1'+k_2V_2')\cdot(t_1V_1'+t_2V_2')-(q+k_1v_1+k_2v_2)\cdot(t_1v_1+t_2v_2)\right]} && \text{for M}' \,. \end{aligned}$$

By using the constraints (3.4) and the properties of an integral lattice, $p \cdot \Delta V_i \in \mathbb{Z}$ for $p, \Delta V_i \in \Lambda$, the mismatch between the phases can be simplified to

$$\Phi' \;=\; \Phi\, e^{-2\pi \mathrm{i}\,(k_1 t_2 - k_2 t_1) V_2 \cdot \Delta V_1} \,. \tag{3.9}$$

That is, the transformation phase of states in model M$'$ differs from the transformation phase of states in model M by a relative phase

$$\widetilde{\varepsilon} \;=\; e^{-2\pi \mathrm{i}(k_1 t_2 - k_2 t_1) V_2 \cdot \Delta V_1} \,. \tag{3.10}$$

According to the nomenclature 'brother models', the relative phase $\widetilde{\varepsilon}$ will be referred to as 'brother phase'. It is straightforward to see that the same relative phase occurs for oscillator states, and the derivation can be repeated for shifts satisfying (2.7) rather than (3.3), yielding the same qualitative result.

The (brother) phase $\widetilde{\varepsilon}$ has certain properties and the fact that it can be non-trivial has important consequences. First of all, $\widetilde{\varepsilon}$ depends on the definition of the model M′, i.e. on the lattice vectors $(\Delta V_1, \Delta V_2)$. Furthermore, it clearly depends on the constructing element g and on the commuting element h,

$$\widetilde{\varepsilon} \;=\; \widetilde{\varepsilon}(g,h)\;. \tag{3.11}$$

It follows from the construction that the brother phase vanishes for $g = (\mathbb{1}, 0)$, i.e. for the untwisted sector. Thus the gauge group and the untwisted sector coincide for brother models. On the other hand, since the brother phase does not vanish in general, the brother models M and M′ may have different twisted sectors, and therefore be inequivalent. This result extends also to the case where we subject the shifts only to the weaker constraints (2.7).

A $\mathbb{Z}_3 \times \mathbb{Z}_3$ example. Let us now study an example to illustrate the results obtained so far. Consider a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold of $E_8 \times E_8$ with standard embedding [9], i.e. model M is defined by

$$V_1 \;=\; \frac{1}{3}\left(1,0,-1,0^5\right)\left(0^8\right) \quad \text{and} \quad V_2 \;=\; \frac{1}{3}\left(0,1,-1,0^5\right)\left(0^8\right)\;. \tag{3.12}$$

The resulting model has an $E_6 \times U(1)^2 \times E_8$ gauge group, 84 $(\overline{\mathbf{27}},\mathbf{1})$ and 243 non-abelian singlets with non-zero U(1) charges.[2] Now define the brother model M′ by

$$\Delta V_1 \;=\; \left(0,-1,0,1,0^4\right)\left(0^8\right) \quad \text{and} \quad \Delta V_2 \;=\; \left(1,0,0,0,1,0^3\right)\left(0^8\right)\;, \tag{3.13}$$

which fulfill the conditions (3.4). From equation (3.10) we find the following non-trivial brother phase

$$\widetilde{\varepsilon}(g,h) \;=\; \widetilde{\varepsilon}(\theta^{k_1}\omega^{k_2},\theta^{t_1}\omega^{t_2}) \;=\; e^{\frac{2\pi i}{3}\,(k_1\,t_2 - k_2\,t_1)}. \tag{3.14}$$

As expected, the gauge group and the untwisted matter of model M′ remain the same as in model M. However, the twisted sectors get modified. The total number of generations is reduced to 3 $(\overline{\mathbf{27}},\mathbf{1})$ and 27 $(\mathbf{27},\mathbf{1})$. The number of singlets remains the same as before, but their localization properties change.

Model M′ is not an unknown construction, but has been studied in the literature in the context of $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifolds with discrete torsion [4]. As we shall see, the brother phase, equation (3.14), is nothing but the discrete torsion phase (equation (4) in ref. [4]). To make this statement more precise, we briefly review discrete torsion in section 3.2, and analyze its relation to the brother phase in section 3.3.

[2] There are three additional singlets $|q\rangle_R \otimes \widetilde{\alpha}^i_{-1}|0\rangle_L$ from the 10d SUGRA multiplet. All orbifold spectra are computed using [14].

3.2 Discrete torsion phase for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds

Let us start with a brief review of discrete torsion in orbifolds, following Vafa [3]. The one-loop partition function Z for a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold has the overall structure

$$Z \;=\; \sum_{\substack{g,h \\ [g,h]=0}} \varepsilon(g,h)\, Z(g,h) \;, \tag{3.15}$$

where the sum runs over pairs of commuting space group elements $g, h \in \mathrm{S}$ and the $\varepsilon(g,h)$ are relative phases between the different terms in the partition function and thus between the different sectors. Different assignments of phases lead, in general, to different orbifold models.

Modular invariance strongly constrains the torsion phases [3]:

$$\varepsilon(g_1 g_2, g_3) \;=\; \varepsilon(g_1, g_3)\, \varepsilon(g_2, g_3) \;, \tag{3.16a}$$
$$\varepsilon(g_1, g_2) \;=\; \varepsilon(g_2, g_1)^{-1} \;. \tag{3.16b}$$

Further, we use the convention

$$\varepsilon(g,g) \;=\; 1 \;. \tag{3.16c}$$

At two-loop, the partition function allows to switch on analogous phases, $\varepsilon(g_1, h_1; g_2, h_2)$. From the requirement of factorizability of the two-loop partition function one infers [3]

$$\varepsilon(g_1, h_1; g_2, h_2) \;=\; \varepsilon(g_1, h_1)\, \varepsilon(g_2, h_2) \;. \tag{3.16d}$$

Following the discussion of ref. [4], in orbifolds without Wilson lines g, h are chosen to be elements of the point group P. In $\mathbb{Z}_N$ orbifolds, due to this choice and equations (3.16) the phases have to be trivial,

$$\varepsilon(g,h) \;=\; 1 \qquad \forall g, h \in P \;. \tag{3.17}$$

Therefore, in the case of $\mathbb{Z}_N$ orbifolds without Wilson lines, non-trivial discrete torsion cannot be introduced.

In $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, still without Wilson lines, the situation is different because there are independent pairs of elements (such that the first element is not a power of the second) which commute with each other. If we take two point group elements $g = \theta^{k_1}\omega^{k_2}$ and $h = \theta^{t_1}\omega^{t_2}$, the equations (3.16) determine the shape of the corresponding phase,

$$\varepsilon(g,h) \;=\; \varepsilon(\theta^{k_1}\omega^{k_2}, \theta^{t_1}\omega^{t_2}) \;=\; e^{\frac{2\pi\mathrm{i}\, m}{M}(k_1 t_2 - k_2 t_1)} \;, \tag{3.18}$$

where $m \in \mathbb{Z}$ [4]. In particular, there are only M inequivalent assignments of ε.

The most important consequence of non-trivial ε-phases for our discussion is that they modify the boundary conditions for twisted states and thus change the twisted spectrum. This can be seen from the transformation phase of equation (2.13), which is modified in the presence of discrete torsion according to

$$\Phi \longmapsto \varepsilon(g,h)\, \Phi \;. \tag{3.19}$$

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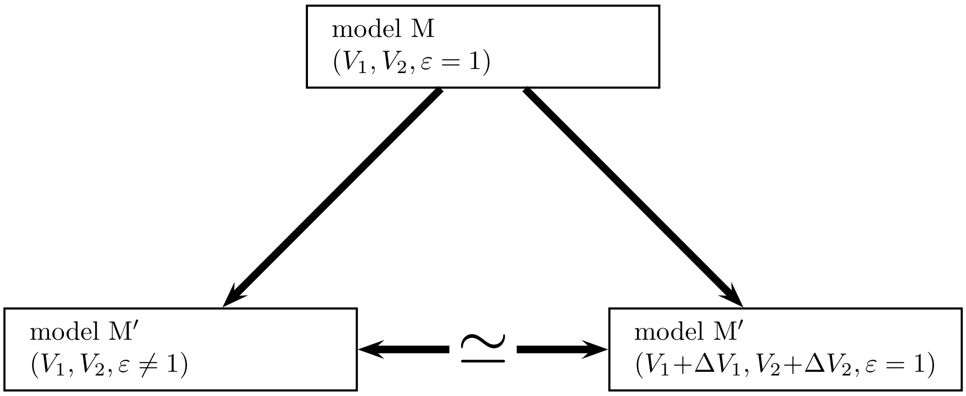

Figure 1: Models with non-trivial discrete torsion have an equivalent description as models with trivial discrete torsion but a different gauge embedding.

3.3 Brother models versus discrete torsion

Let us now come back to the task of establishing the relation between the discrete torsion phase and the brother phase as introduced in section 3.1. From equations (3.10) and (3.18) it is clear that both phases can be made to coincide. More precisely, since V_2 can be written as $V_2 = \frac{\lambda_2}{M}$ with $\lambda_2 \in \Lambda$ (cf. equation (2.5)), one can achieve

$$-V_2 \cdot \Delta V_1 \;=\; \frac{m}{M} \tag{3.20}$$

for an appropriate choice of $\Delta V_1 \in \Lambda$. Since the solutions to the mass equations and the projection conditions are the same in a model with discrete torsion and a brother model, whose associated phases fulfill equation (3.20), the spectra of both models coincide. We will therefore regard both models as *equivalent*. This means that introducing a discrete torsion phase, equation (3.18), is equivalent to changing the gauge embedding according to

$$(V_1, V_2) \;\rightarrow\; (V_1 + \Delta V_1, V_2 + \Delta V_2) \tag{3.21}$$

with $\Delta V_i \in \Lambda$ and $-V_2 \cdot \Delta V_1 = m/M$. In particular, the assignment of discrete torsion to a given $\mathbb{Z}_N \times \mathbb{Z}_M$ model is a 'gauge-dependent' statement in the sense that torsion can be traded for changing the gauge embedding (cf. figure 1).

To illustrate our result, we construct the standard embedding models for $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds with an $E_8 \times E_8$ lattice of ref. [4] with discrete torsion in terms of non-standard embedding shifts without discrete torsion (brother models). We use the following recipe to construct brother models, i.e. mimic models with discrete torsion:
For a given set of shifts V_1 and V_2 fulfilling the modular invariance conditions, find a new set of shifts $V_1' = V_1 + \Delta V_1$ and $V_2' = V_2 + \Delta V_2$ with the following properties:

(i) the new shifts differ from the original set only by lattice vectors, i.e. $\Delta V_1, \Delta V_2 \in \Lambda$

(ii) the new shifts also fulfill the modular invariance conditions, and

(iii) the 'interference term' $V_2 \cdot \Delta V_1$ is not an integer.

orbifold	torsion ε	shift V_1	shift V_2
$\mathbb{Z}_2 \times \mathbb{Z}_2$	1	$\left(\frac{1}{2}, 0, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$
	-1	$\left(\frac{1}{2}, -1, -\frac{1}{2}, 1, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0, 0\right)$
$\mathbb{Z}_4 \times \mathbb{Z}_2$	1	$\left(\frac{1}{4}, 0, -\frac{1}{4}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$
	-1	$\left(\frac{1}{4}, -1, -\frac{1}{4}, 1, 0, 0, 0, 0\right)$	$\left(2, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$
$\mathbb{Z}_6 \times \mathbb{Z}_2$	1	$\left(\frac{1}{6}, 0, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{2}, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$
	-1	$\left(\frac{1}{6}, -1, -\frac{1}{6}, 1, 0, 0, 0, 0\right)$	$\left(3, \frac{1}{2}, -\frac{1}{2}, 0, 1, 0, 0, 0\right)$
$\mathbb{Z}_6' \times \mathbb{Z}_2$	1	$\left(\frac{1}{6}, \frac{1}{6}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$	$\left(\frac{1}{2}, 0, -\frac{1}{2}, 0, 0, 0, 0, 0\right)$
	-1	$\left(-\frac{5}{6}, \frac{7}{6}, -\frac{1}{3}, 1, 1, 0, 0, 0\right)$	$\left(\frac{1}{2}, 3, -\frac{1}{2}, 1, 0, 0, 0, 0\right)$
$\mathbb{Z}_3 \times \mathbb{Z}_3$	1	$\left(\frac{1}{3}, 0, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$
	$e^{2\pi i\frac{1}{3}}$	$\left(\frac{1}{3}, -1, -\frac{1}{3}, 1, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{3}, -\frac{1}{3}, 0, 1, 0, 0, 0\right)$
	$e^{2\pi i\frac{2}{3}}$	$\left(\frac{1}{3}, -2, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$	$\left(2, \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$
$\mathbb{Z}_6 \times \mathbb{Z}_3$	1	$\left(\frac{1}{6}, 0, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$
	$e^{2\pi i\frac{1}{3}}$	$\left(\frac{1}{6}, -1, -\frac{1}{6}, 1, 0, 0, 0, 0\right)$	$\left(2, \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$
	$e^{2\pi i\frac{2}{3}}$	$\left(\frac{1}{6}, -2, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(4, \frac{1}{3}, -\frac{1}{3}, 0, 0, 0, 0, 0\right)$
$\mathbb{Z}_4 \times \mathbb{Z}_4$	1	$\left(\frac{1}{4}, 0, -\frac{1}{4}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0\right)$
	i	$\left(\frac{1}{4}, -1, -\frac{1}{4}, 1, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{4}, -\frac{1}{4}, 0, 1, 0, 0, 0\right)$
	-1	$\left(\frac{1}{4}, -2, -\frac{1}{4}, 0, 0, 0, 0, 0\right)$	$\left(2, \frac{1}{4}, -\frac{1}{4}, 0, 0, 0, 0, 0\right)$
	$-i$	$\left(\frac{1}{4}, -3, -\frac{1}{4}, 1, 0, 0, 0, 0\right)$	$\left(3, \frac{1}{4}, -\frac{1}{4}, 0, 1, 0, 0, 0\right)$
$\mathbb{Z}_6 \times \mathbb{Z}_6$	1	$\left(\frac{1}{6}, 0, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(0, \frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$
	$e^{2\pi i\frac{1}{6}}$	$\left(\frac{1}{6}, -1, -\frac{1}{6}, 1, 0, 0, 0, 0\right)$	$\left(1, \frac{1}{6}, -\frac{1}{6}, 0, 1, 0, 0, 0\right)$
	$e^{2\pi i\frac{1}{3}}$	$\left(\frac{1}{6}, -2, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(2, \frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$
	-1	$\left(\frac{1}{6}, -3, -\frac{1}{6}, 1, 0, 0, 0, 0\right)$	$\left(3, \frac{1}{6}, -\frac{1}{6}, 0, 1, 0, 0, 0\right)$
	$e^{2\pi i\frac{2}{3}}$	$\left(\frac{1}{6}, -4, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$	$\left(4, \frac{1}{6}, -\frac{1}{6}, 0, 0, 0, 0, 0\right)$
	$e^{2\pi i\frac{5}{6}}$	$\left(\frac{1}{6}, -5, -\frac{1}{6}, 1, 0, 0, 0, 0\right)$	$\left(5, \frac{1}{6}, -\frac{1}{6}, 0, 1, 0, 0, 0\right)$

Table 1: $\mathbb{Z}_N \times \mathbb{Z}_M$ models with discrete torsion and standard embedding are equivalent to models without discrete torsion and non-standard embedding. We write the torsion phase factor as $\varepsilon = e^{-2\pi i\, V_2 \cdot \Delta V_1}$. The components of the shifts within the second E_8 all vanish. This result also applies to orbifold models in SO(32).

In practice (and for any N, M), the above properties can be expressed in terms of linear Diophantine equations for which we always find solutions.

Possible choices for the shifts $(V_1 + \Delta V_1, V_2 + \Delta V_2)$ are shown in table 1, where we list the shifts of torsionless models equivalent to the discrete torsion model of ref. [4].

Our result has important consequences for the classification of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds.

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Introducing a discrete torsion phase in the sense of ref. [4] does not lead to new models. That is, all models with this discrete torsion can be equivalently obtained by scanning over torsionless models only. This will be important for our classification in section 5.

It is also instructive to interpret the equivalence between discrete torsion and changing the gauge embedding in terms of geometry. Discrete torsion can be regarded as a property of the 6D compact space while changing the gauge embedding affects the (left-moving) coordinates of the gauge lattice only. Hence one might argue that discrete torsion and choosing a different gauge embedding are two different features of orthogonal dimensions. However, by embedding the 'spatial' twist in the gauge degrees of freedom, these features get combined in such a way that it is no longer possible to make a clear separation. Using a more technical language one might rephrase this statement by saying that, since physical states arise from tensoring left- and right-movers together, the phases ε and $\widetilde{\varepsilon}$ cannot be distinguished. Consequently, properties of the zero-modes cannot be ascribed neither to the gauge embedding alone nor to the presence of discrete torsion, but only to both.

4. Generalized discrete torsion

The results of the previous section can be generalized. To see this, we first generalize the brother phase of section 3.1 for orbifolds with Wilson lines. In a second step, we compare the emerging phases to what is known as generalized discrete torsion [7]. As before, we can relate both phases.

4.1 Generalized brother models

Let us turn to the discussion of orbifolds with Wilson lines [15]. A (torsionless) model M is defined by (V_1, V_2, A_α). A brother model M′ appears by adding lattice vectors to the shifts and Wilson lines, i.e. M′ is defined by

$$(V_1', V_2', A_\alpha') \;=\; (V_1 + \Delta V_1, V_2 + \Delta V_2, A_\alpha + \Delta A_\alpha)\;, \tag{4.1}$$

with $\Delta V_i, \Delta A_\alpha \in \Lambda$. From the conditions (2.7), the choice of lattice vectors $(\Delta V_i, \Delta A_\alpha)$ is constrained by

$$M\,(V_1 \cdot \Delta V_2 + V_2 \cdot \Delta V_1 + \Delta V_1 \cdot \Delta V_2) = 0 \bmod 2 \;\equiv\; 2\,x\;, \tag{4.2a}$$

$$N_\alpha\,(V_i \cdot \Delta A_\alpha + A_\alpha \cdot \Delta V_i + \Delta V_i \cdot \Delta A_\alpha) = 0 \bmod 2 \;\equiv\; 2\,y_{i\alpha}\;, \tag{4.2b}$$

$$Q_{\alpha\beta}\,(A_\alpha \cdot \Delta A_\beta + A_\beta \cdot \Delta A_\alpha + \Delta A_\alpha \cdot \Delta A_\beta) = 0 \bmod 2 \;\equiv\; 2\,z_{\alpha\beta}\;, \tag{4.2c}$$

where x, $y_{i\alpha}$, $z_{\alpha\beta} \in \mathbb{Z}$.

Repeating the steps of section 3.1 one arrives at a 'generalized brother phase'

$$\begin{aligned}\widetilde{\varepsilon} = \exp\Big\{-2\pi\mathrm{i}\,\Big[&(k_1\,t_2 - k_2\,t_1)\Big(V_2\cdot\Delta V_1 - \frac{x}{M}\Big)\\ &+(k_1\,m_\alpha - t_1\,n_\alpha)\Big(A_\alpha\cdot\Delta V_1 - \frac{y_{1\alpha}}{N_\alpha}\Big)\\ &+(k_2\,m_\alpha - t_2\,n_\alpha)\Big(A_\alpha\cdot\Delta V_2 - \frac{y_{2\alpha}}{N_\alpha}\Big)\\ &+n_\alpha\,m_\beta\Big(A_\beta\cdot\Delta A_\alpha - \frac{z_{\alpha\beta}}{Q_{\alpha\beta}}\Big)\Big]\Big\}\ ,\end{aligned} \tag{4.3}$$

corresponding to the constructing element $g = (\theta^{k_1}\omega^{k_2}, n_\alpha e_\alpha)$ and the commuting element $h = (\theta^{t_1}\omega^{t_2}, m_\alpha e_\alpha)$. One can see that $D_{\alpha\beta} \equiv A_\beta\cdot\Delta A_\alpha - z_{\alpha\beta}/Q_{\alpha\beta}$ is (almost) antisymmetric in α, β,

$$D_{\alpha\beta} = -D_{\beta\alpha} \bmod 1\,. \tag{4.4}$$

Notice that also in the case of orbifolds with lattice-valued Wilson lines, $A_\alpha \in \Lambda$, the last three terms of equation (4.3) can be non-trivial, giving rise to new brother models.

Brother models in $\mathbb{Z}_N$ orbifolds. From equation (4.3), it is clear that the generalized brother phase is also important for $\mathbb{Z}_N$ orbifolds. More precisely, in $\mathbb{Z}_N$ orbifolds with Wilson lines, the second and fourth lines of equation (4.3) are not always trivial and thus also lead to brother models.

Let us illustrate this with an example in $\mathbb{Z}_4$ with twist $v = \frac{1}{4}(-2,\,1,\,1;\,0)$ acting on the compactification lattice $\Gamma = \mathrm{SO}(4)^3$, and standard embedding [16]. The gauge group is $\mathrm{E}_6{\times}\mathrm{SU}(2){\times}\mathrm{E}_8$. By turning on the lattice-valued Wilson lines

$$A_1 = \left(0^8\right)\left(1^2, 0^6\right), \qquad A_5 = A_6 = \left(0^8\right)\left(0, 1^2, 0^5\right), \tag{4.5}$$

a non-trivial generalized brother phase with $D_{15} = D_{16} = -\frac{1}{2}$ is introduced. The untwisted and first twisted sectors remain unchanged, but the number of (anti-) families in the second twisted sector is reduced from 10 $(\overline{\mathbf{27}},\mathbf{1},\mathbf{1})$ + 6 $(\mathbf{27},\mathbf{1},\mathbf{1})$ to 6 $(\overline{\mathbf{27}},\mathbf{1},\mathbf{1})$ + 2 $(\mathbf{27},\mathbf{1},\mathbf{1})$.

4.2 Generalized discrete torsion

In section 3.2 we have discussed the discrete torsion phase as introduced in ref. [4]. More recently, this concept has been extended by introducing a generalized discrete torsion phase in the context of type IIA/B string theory [7]. This generalized torsion phase depends on the fixed points of the orbifold. It weights differently terms in the partition function corresponding to the same twisted sector but different fixed points, and is constrained by modular invariance.

Following the steps of section 3.2 and considering $g, h \in S$, we write down the general solution of equations (3.16) for the discrete torsion phase as[3]

$$\varepsilon(g,h) \;=\; e^{2\pi\mathrm{i}\,[a\,(k_1\,t_2-k_2\,t_1)+b_\alpha\,(k_1\,m_\alpha-t_1\,n_\alpha)+c_\alpha\,(k_2\,m_\alpha-t_2\,n_\alpha)+d_{\alpha\beta}\,n_\alpha\,m_\beta]}\,. \tag{4.6}$$

[3]Note that we employ the stronger constraints (3.16) rather than the conditions presented in [7]. It might be possible to relax condition (3.16b), in which case additional possibilities could arise. We ignore this possibility in the present study.

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Modular invariance constrains the values of $a, b_\alpha, c_\alpha, d_{\alpha\beta}$. Therefore $a = \widetilde{a}/M$, $b_\alpha = \widetilde{b}_\alpha/N_\alpha$, $c_\alpha = \widetilde{c}_\alpha/N_\alpha$, $d_{\alpha\beta} = \widetilde{d}_{\alpha\beta}/N_{\alpha\beta}$ with $\widetilde{a}$, $\widetilde{b}_\alpha$, $\widetilde{c}_\alpha$, $\widetilde{d}_{\alpha\beta} \in \mathbb{Z}$, $N_{\alpha\beta}$ being the greatest common divisor of N_α and N_β. In addition, $d_{\alpha\beta}$ must be antisymmetric in α, β.

The parameters b_α, c_α, $d_{\alpha\beta}$ are additionally constrained by the geometry of the orbifold. It is not hard to see that if $e_\alpha \simeq e_\beta$ on the orbifold, then $b_\alpha = b_\beta$, $c_\alpha = c_\beta$ and $d_{\alpha\beta} = 0$ must hold (cf. the examples below).

The generalized discrete torsion is not restricted only to $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, as the usual discrete torsion was, but will likewise appear in the $\mathbb{Z}_N$ case. Clearly, since in $\mathbb{Z}_N$ orbifolds there is only one shift, the parameters a and c_α vanish.

Examples. Let us consider the $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold compactified on an $\mathrm{SU}(3)^3$ lattice. In this case we have $e_1 \simeq e_2$, $e_3 \simeq e_4$ and $e_5 \simeq e_6$ on the orbifold. This implies that there are only three independent b_α, namely b_1, b_3, b_5, while $b_2 = b_1$, $b_4 = b_3$, $b_6 = b_5$. Analogously, only c_1, c_3, c_5 are independent. Further, the antisymmetric matrix $d_{\alpha\beta}$ takes the form

$$d_{\alpha\beta} = \begin{pmatrix} 0 & 0 & d_1 & d_1 & d_2 & d_2 \\ 0 & 0 & d_1 & d_1 & d_2 & d_2 \\ -d_1 & -d_1 & 0 & 0 & d_3 & d_3 \\ -d_1 & -d_1 & 0 & 0 & d_3 & d_3 \\ -d_2 & -d_2 & -d_3 & -d_3 & 0 & 0 \\ -d_2 & -d_2 & -d_3 & -d_3 & 0 & 0 \end{pmatrix} . \tag{4.7}$$

Including the parameter a, there are 10 independent discrete torsion parameters, which can take values 0, $\frac{1}{3}$ or $\frac{2}{3}$.

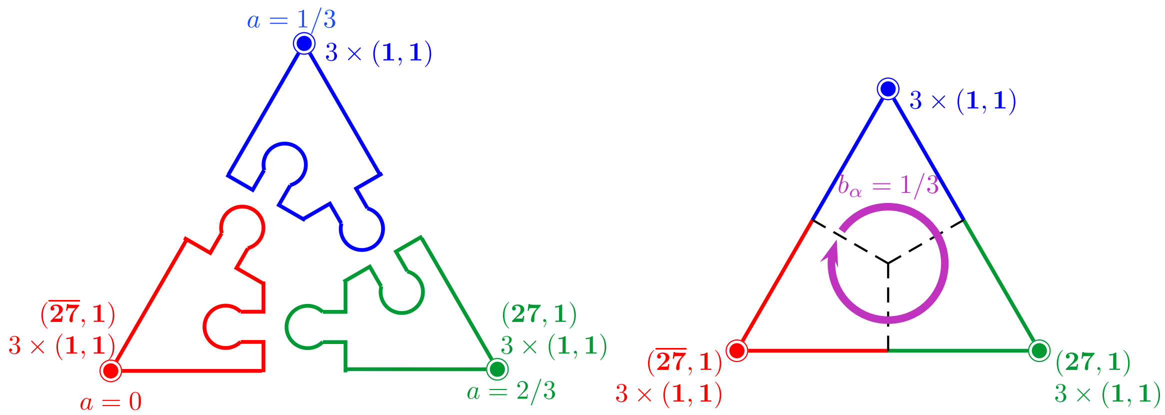
For the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold on an $\mathrm{SU}(2)^6$ lattice an analogous consideration shows that there are no restrictions for the discrete torsion parameters. Therefore, there are $1+6+6+15 = 28$ independent parameters a, b_α, c_α, $d_{\alpha\beta}$, with values either 0 or $\frac{1}{2}$. However, since the coefficients $n_\alpha m_\beta$ of $d_{\alpha\beta}$ for $(\alpha, \beta) \in \{(1,2), (3,4), (5,6)\}$ vanish, the corresponding $d_{\alpha\beta}$ are not physical, leading to 25 effective parameters.

Generalized discrete torsion and local spectra. In order to understand the action of the generalized discrete torsion, let us consider the following example. We start with the $\mathbb{Z}_3 \times \mathbb{Z}_3$ standard embedding without Wilson lines, $A_\alpha = 0$, and switch on the discrete torsion phase, equation (4.6), with $b_3 = b_4 = \frac{1}{3}$. The total number of families is reduced from 84 $(\overline{\mathbf{27}}, \mathbf{1})$ to 24 $(\overline{\mathbf{27}}, \mathbf{1})$ and 12 $(\mathbf{27}, \mathbf{1})$.

Due to its form, the discrete torsion phase $\varepsilon = e^{2\pi\mathrm{i}\, b_\alpha\, (k_1\, m_\alpha - t_1\, n_\alpha)}$ distinguishes between different fixed points of a particular twisted sector. That is, generalized discrete torsion can be thought of as a local feature. In general, the additional phase at a given fixed point coincides with a brother phase of the torsionless model (cf. first term of equation (4.3)), i.e. locally one can find ΔV_i such that

$$\varepsilon = e^{2\pi\mathrm{i}\, b_\alpha\, (k_1\, m_\alpha - t_1\, n_\alpha)} = e^{-2\pi\mathrm{i}\, (k_1\, t_2 - k_2\, t_1)\left(V_2 \cdot \Delta V_1 - \frac{x}{3}\right)} \tag{4.8}$$

with appropriate x. Then, each local spectrum coincides with the local spectrum of some brother model. The interpretation of generalized discrete torsion in terms of 'localized



(a) (b)

Figure 2: Sketch of a (2D) SU(3) plane of a $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold (the second plane in the example). Parts ('corners') from different brother models (a) can be 'sewed together' to a model in which the torsion phase differs for different fixed points. This is equivalent to switching on the generalized discrete torsion phase b_α (b).

discrete torsion' parallels the concept of local shifts (cf. [12, 13]) in orbifolds with Wilson lines.

Note that ΔV_i as in (4.8) cannot be found for twisted sectors where b_α corresponds to a direction e_α of a fixed torus, where b_α projects out all states of the sector.

For concreteness, we first focus on the three fixed points in the second torus of the $T_{(0,1)}$ twisted sector. As depicted in figure 2, the local spectra of the three brother models, $a \equiv -\left(V_2 \cdot \Delta V_1 - \frac{x}{3}\right) = 0, \frac{1}{3}, \frac{2}{3}$, can be combined consistently into one model with $b_3 = b_4 = \frac{1}{3}$. On the other hand, in the $T_{(1,0)}$ twisted sector there is a fixed torus in the directions e_3, e_4; thus the sector is empty.

This procedure can also be applied to the terms c_α and $d_{\alpha\beta}$ of the generalized discrete torsion phase, equation (4.6).

Generalized brother models versus generalized discrete torsion. As in our previous discussion in section 3, also the generalized versions of the discrete torsion phase and the brother phase have a very similar form. Indeed, whenever there are non-trivial solutions to equations (4.2), one can equivalently describe models with generalized discrete torsion phase in terms of generalized brother models. This is the generic case.

However, there are exceptions. Namely, as we will explain below, models with $d_{\alpha\beta} \neq 0$ in $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifolds without Wilson lines cannot be interpreted in terms of brother models.

Consider the fourth part of the generalized discrete torsion phase of equation (4.6),

$$\varepsilon \;=\; e^{2\pi \mathrm{i}\, d_{\alpha\beta}\, n_\alpha m_\beta}\,, \tag{4.9}$$

with $d_{\alpha\beta} \in \left\{0, \frac{1}{3}, \frac{2}{3}\right\}$. An analogous term appears in the generalized brother phase as

$$\widetilde{\varepsilon} \;=\; \exp\left[-2\pi \mathrm{i}\, n_\alpha m_\beta \left(A_\beta \cdot \Delta A_\alpha - \frac{z_{\alpha\beta}}{Q_{\alpha\beta}}\right)\right]\,, \tag{4.10}$$

where $Q_{\alpha\beta} = 3$, since the Wilson lines have order 3. In general, both phases can be made coincide by choosing $\Delta A_\alpha \in \Lambda$ such that

$$-\left(A_\beta \cdot \Delta A_\alpha - \frac{z_{\alpha\beta}}{3}\right) = d_{\alpha\beta}\ . \tag{4.11}$$

On the other hand, in the case when $A_\alpha = 0$ and $\Delta A_\alpha \neq 0$, equation (4.10) simplifies to

$$\widetilde{\varepsilon} \;=\; e^{2\pi\mathrm{i}\, n_\alpha m_\beta \left(\frac{z_{\alpha\beta}}{3}\right)} \;=\; e^{2\pi\mathrm{i}\, n_\alpha m_\beta \left(\frac{\Delta A_\alpha \cdot \Delta A_\beta}{2}\right)}\ , \tag{4.12}$$

where the second equality follows from the definition of $z_{\alpha\beta}$, equation (4.2c). As ΔA_α are lattice vectors, this equality can only hold if $z_{\alpha\beta} = 0 \bmod 3$, which implies that the brother phase equation (4.12) is trivial. Thus, in this case, the generalized discrete torsion phase leads to models which cannot arise by adding lattice vectors to shifts and Wilson lines.

In summary, the generalized discrete torsion phases admit more possible assignments than the generalized brother phases. Nevertheless, a large class of the models with generalized discrete torsion has an equivalent description in terms of models with a modified gauge embedding.

Our results have important implications. By introducing generalized discrete torsion, or lattice-valued Wilson lines, one can control the local spectra. We therefore expect that introducing generalized discrete torsion, or alternatively shifting the Wilson lines by lattice vectors, will gain a similar importance as discrete Wilson lines [15] for orbifold model building.

As stated above, switching on generalized discrete torsion can lead to the disappearance of complete local spectra. This raises the question of how to interpret this fact in terms of geometry. Some of the localized zero-modes can be viewed as blow-up modes which allow to resolve the orbifold singularity associated to a given fixed point [17, 18, 5] (see [19 – 21] for recent developments). If at a given fixed point there are no zero modes, one might argue that, therefore, the associated singularity cannot be 'blown up'. In what follows, we shall advertise an alternative interpretation.

4.3 Connection to non-factorizable orbifolds

We find that in many cases orbifold models M with certain geometry, i.e. compactification lattice Γ, and generalized discrete torsion switched on are equivalent to torsionless models M′ based on a different lattice Γ'. Model M′ has less fixed points than M, and the mismatch turns out to constitute precisely the 'empty' fixed points of model M.

The simplest examples are based on $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds with standard embedding and without Wilson lines. As compactification lattice Γ, we choose an $\mathrm{SU}(2)^6$ lattice [10]. As we have seen in section 4.2, in this case there are 25 physical parameters for generalized discrete torsion, with values either 0 or $\frac{1}{2}$. For concreteness, we restrict to the 12 $d_{\alpha\beta}$ parameters and scan over all 2^{12} models.

Beside other models with a net number of zero families, we find eight models (and their mirrors, i.e. models where families and anti-families are exchanged). They are listed in table 2, where we present the number of (anti-)families for each twisted sector and the total number of singlets. As discussed in section 4.2, models with non-trivial $d_{\alpha\beta}$ are

	$T_{(1,0)}$	$T_{(0,1)}$	$T_{(1,1)}$	total	$\#_S$	$d_{\alpha\beta} = \frac{1}{2}$	$A_\alpha \neq 0$
A.1	$(16,0)$	$(16,0)$	$(16,0)$	$(51,3)$	246	–	–
A.2	$(12,4)$	$(8,0)$	$(8,0)$	$(31,7)$	166	d_{24}	$A_2 = (S)(0^8)$, $A_4 = (V)(0^8)$
A.3	$(10,6)$	$(4,0)$	$(4,0)$	$(21,9)$	126	d_{14}, d_{23}	$A_1 = (S)(0^8)$, $A_2 = (0^8)(S)$, $A_3 = (0^8)(V)$, $A_4 = (V)(0^8)$
A.4	$(8,0)$	$(8,0)$	$(8,0)$	$(27,3)$	126	d_{26}, d_{46}	$A_2 = (V')(0^8)$, $A_4 = (V)(0^8)$, $A_6 = (S)(0^8)$
A.5	$(6,2)$	$(6,2)$	$(4,0)$	$(19,7)$	106	d_{24}, d_{36}	$A_2 = (S)(0^8)$, $A_3 = (0^8)(S)$, $A_4 = (V)(0^8)$, $A_6 = (0^8)(V)$
A.6	$(6,2)$	$(4,0)$	$(4,0)$	$(17,5)$	86	d_{16}, d_{24}, d_{36}	$A_1 = (V)(0^8)$, $A_2 = (0^8)(V)$, $A_3 = (V')(0^8)$, $A_4 = (0^8)(S)$, $A_6 = (S)(0^8)$
A.7	$(4,0)$	$(4,0)$	$(4,0)$	$(15,3)$	66	d_{16}, d_{25}, d_{36}, d_{45}	$A_1 = (V)(0^8)$, $A_2 = (0^8)(V)$, $A_3 = (V')(0^8)$, $A_4 = (0^8)(V')$, $A_5 = (0^8)(S)$, $A_6 = (S)(0^8)$
A.8	$(3,1)$	$(3,1)$	$(3,1)$	$(12,6)$	66	d_{16}, d_{24}, d_{35}	$A_1 = (W_1)(0^8)$, $A_2 = (0^8)(W_1)$, $A_3 = (0^8)(W_1')$, $A_4 = (0^8)(W_2)$, $A_5 = (0^8)(W_2')$, $A_6 = (W_2)(0^8)$

Table 2: Survey of $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds with generalized discrete torsion. The 2^{nd}–4^{th} columns list the number of anti-families and families, respectively, for the various twisted sectors $T_{(k_1,k_2)}$. In all models, the untwisted sector gives a contribution of $(3,3)$ (anti-)families. $\#_S$ denotes the total number of singlets. These spectra can either be obtained by turning on generalized discrete torsion $d_{\alpha\beta}$ as specified in the next-to-last column, or by using lattice-valued Wilson lines A_α as listed in the last column. The building blocks are defined in equation (4.13).

equivalent to torsionless models with lattice-valued Wilson lines. Possible representatives of these Wilson lines can be composed out of the building blocks

$$\begin{aligned} W_1 &= (0^6,1,1)\,, \quad & W_2 &= (0^5,1,1,0)\,, \quad & W_1' &= (1,1,0^6)\,, \quad & W_2' &= (0,1,1,0^5)\,, \\ S &= (\tfrac{1}{2}^8)\,, & V &= (0^7,2)\,, & V' &= (0^6,2,0)\,, \end{aligned} \tag{4.13}$$

and are listed in the last column of table 2.

Models leading to spectra coinciding with what we got in table 2 have already been discussed in the literature. They appeared first in ref. [22] in the context of free fermionic string models related to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold with an additional freely acting shift. More recently, new $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold constructions have been found in studying orbifolds of non-factorizable six-tori [23, 24]. We find that for each model M of table 2 there is a corresponding 'non-factorizable' model M′ with the following properties:

(i) Each 'non-empty' fixed point, i.e. each fixed point with local zero-modes, in the model M can be mapped to a fixed point with the same spectrum in model M′.

(ii) The number of 'non-empty' fixed points in M coincides with the total number of fixed points in M′.

These relations are not limited to $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolds, rather we find an analogous connection also in other $\mathbb{Z}_N \times \mathbb{Z}_M$ cases ($\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds based on non-factorizable compactification lattices have recently been discussed in [25]). This result hints towards an intriguing impact of generalized discrete torsion on the interpretation of orbifold geometry. What the (zero-mode) spectra concerns, introducing generalized discrete torsion (or considering generalized brother models) is equivalent to changing the geometry of the underlying compact space, $\Gamma \to \Gamma'$. To establish complete equivalence between these models would require to prove that the couplings of the corresponding states are the same, which is beyond the scope of the present study. It is, however, tempting to speculate that non-resolvable singularities, as discussed above, do not 'really' exist as one can always choose (for a given spectrum) the compactification lattice Γ in such a way that there are no 'empty' fixed points.

5. How to classify $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds

Let us now turn to describing a method of classifying heterotic $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, taking into account generalized discrete torsion. To illustrate our methods, we focus on $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifold compactifications of the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string. It is straightforward to generalize the discussion to other $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds and to the SO(32) case.

To classify an orbifold requires an efficient prescription of how to obtain all inequivalent models. The first step in a classification is to get all admissible choices for the shift vector V_1. For this purpose, we make use of Dynkin diagram techniques (see e.g. [16]). These techniques are advantageous since, when writing down V_1, one has the freedom of choosing the basis of the weight lattice Λ in such a way that the shift has a very simple form. Clearly, this freedom is lost when one introduces the second shift (and Wilson lines). This complicates the construction of all inequivalent shifts V_2.

To obtain all inequivalent V_2 we utilize a method introduced by Giedt [26] (see also [27]), i.e. use an adequate minimal ansatz which avoids redundancies due to lattice translations and some Weyl reflections. This ansatz restricts the shifts to be only in a certain cell $\widetilde{\Lambda}_N$ of the lattice Λ in such a way that any possible $\mathbb{Z}_N$ shift can be written as an element of this cell plus a lattice vector. That is, an arbitrary $\mathbb{Z}_N$ shift has a unique decomposition

$$V \;=\; \widetilde{V} + \Delta V\;, \quad \text{where } \widetilde{V} \in \widetilde{\Lambda}_N \text{ and } \Delta V \in \Lambda\;. \tag{5.1}$$

Consider now a consistent gauge embedding (V_1, V_2). According to equation (5.1) the shifts can be decomposed into $(\widetilde{V}_1 + \Delta V_1, \widetilde{V}_2 + \Delta V_2)$ with $\widetilde{V}_1 \in \widetilde{\Lambda}_N$, $\widetilde{V}_2 \in \widetilde{\Lambda}_M$ and $\Delta V_i \in \Lambda$. It is not hard to see that the conditions (2.7) imply

$$N\left(\widetilde{V}_1^2 - v_1^2\right) \;=\; 0 \bmod 2\;, \tag{5.2a}$$

$$M\left(\widetilde{V}_2^2 - v_2^2\right) \;=\; 0 \bmod 2\;, \tag{5.2b}$$

$$M\left(\widetilde{V}_1 \cdot \widetilde{V}_2 - v_1 \cdot v_2\right) \;=\; 0 \bmod 1\;. \tag{5.2c}$$

To scan over all possible shift embeddings is therefore reduced to the task of

(i) specifying $\widetilde{V}_1$ (satisfying (5.2a)) by Dynkin techniques,

(ii) scanning $\widetilde{\Lambda}_M$ for $\widetilde{V}_2$ fulfilling equations (5.2b) and (5.2c), and

(iii) examining all possible (V_1, V_2) related to $(\widetilde{V}_1, \widetilde{V}_2)$ by lattice translations and satisfying (2.7).

At first sight, the last step seems to require a scan over an infinite number of shifts. However, as we have seen in section 3, given one representative (V_1, V_2) satisfying (2.7), this scan can be replaced by switching on all inequivalent discrete torsion phases. Since there is only a finite number of torsion phase assignments, we have found a prescription to obtain all inequivalent models by scanning only over a finite set of inputs.

As already stated, in a complete classification it is necessary to take into account generalized discrete torsion as well. Thus, to get all admissible shifts, one has to scan over all appropriate values for the parameters b_α, c_α and $d_{\alpha\beta}$.

All statements made for V_2 apply also to the Wilson lines. That is, in order to obtain all inequivalent Wilson line embeddings, one can also scan the finite cell $\widetilde{\Lambda}_{N_\alpha}$ (fulfilling consistency conditions analogous to (5.2)), and then switch on generalized discrete torsion.

So far, we have described how to obtain all inequivalent models. However, some of the inputs, specified by (V_1, V_2), Wilson lines and generalized torsion phases, turn out to be equivalent. Apart from the ambiguities related to Weyl reflections (cf. [26]), in the framework of $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds further complications arise. For instance, we note that models with shifts related by discrete rotations, i.e. $(V_1, V_2) \to (a_1\, V_1 + a_2\, V_2, b_1\, V_1 + b_2\, V_2)$ with proper values of $a_i, b_i \in \mathbb{Z}$, can be equivalent. For example, in $\mathbb{Z}_3 \times \mathbb{Z}_3$ a model with shifts (V_1, V_2) is equivalent to a model with shifts $(V_1, V_1 + 2\, V_2)$.

In our classification below, we consider two models as inequivalent if and only if their massless spectra are different.[4]

Sample classification of $\mathbb{Z}_3 \times \mathbb{Z}_3$ without Wilson lines. As a concrete application, let us describe the classification of $\mathbb{Z}_3 \times \mathbb{Z}_3$ orbifolds without Wilson lines.[5] By using Dynkin diagram techniques, one finds that there are only five consistent shift vectors V_1, which can be written in the generic form

$$\widetilde{V}_1 \;=\; \frac{1}{3}(0^{n_0}, 1^{n_1}, 2^{\alpha})\,(0^{m_0}, 1^{m_1}, 2^{\beta}), \tag{5.3}$$

where α and β can be either 0 or 1, and $n_0, n_1, m_0, m_1 \in \mathbb{Z}$, such that $n_0 + n_1 + \alpha = m_0 + m_1 + \beta = 8$.

[4] In practice, we compare the non-Abelian massless spectra and the number of singlets. This underestimates the true number of models somewhat.

[5] As we allow for generalized discrete torsion, some of the models can be interpreted as being endowed with lattice-valued Wilson lines, see section 4.

A general ansatz describing the second shift V_2 of order three is given by [26]:

$$\widetilde{V}_2 = \frac{1}{3}\left(\left(\begin{array}{c}3\\ \vdots\\ -2\end{array}\right),\left(\begin{array}{c}1\\ 0\end{array}\right)^{n_0-1},\left(\begin{array}{c}1\\ 0\\ -1\end{array}\right)^{n_1+\alpha}\right)$$

$$\left(\left(\begin{array}{c}3\\ \vdots\\ -2\end{array}\right),\left(\begin{array}{c}1\\ 0\end{array}\right)^{m_0-1},\left(\begin{array}{c}1\\ 0\\ -1\end{array}\right)^{m_1+\beta}\right), \tag{5.4}$$

subject to lattice conditions, equations (2.5), and to the necessary conditions (5.2). Some of these models do not fulfill the consistency requirements (2.7). As explained above, in these cases we proceed by identifying lattice vectors ΔV_i with the property that $(\widetilde{V}_1 + \Delta V_1, \widetilde{V}_2 + \Delta V_2)$ satisfy (2.7). The problem of finding those lattice vectors can be reduced to a set of linear Diophantine equations.

To generate all shift embeddings, we compute the spectra of models with different values of a in the discrete torsion phase. In $\mathbb{Z}_3 \times \mathbb{Z}_3$, the parameter a can have values 0, $\frac{1}{3}$ or $\frac{2}{3}$. This gives a factor of three to the total number of models. However, it turns out that not all of them are inequivalent. Counting only inequivalent spectra, we find that there are 120 inequivalent shift embeddings.

We use now the set of shift embeddings to generate all admissible models. As discussed in the examples of section 4.2, excluding a, there are 9 independent generalized discrete torsion parameters, whose values can be again 0, $\frac{1}{3}$ or $\frac{2}{3}$. Although the number of models is multiplied by a factor 3^9, the number of all inequivalent models (spectra) is 1082. These models comprise the complete set of admissible models without Wilson lines, or, more precisely, the complete set of models which can be described by vanishing Wilson lines. The model definitions and the resulting spectra are given in [28].

The above procedure can straightforwardly be carried over to the SO(32) case. The results turn out to be similar. Repeating the steps of our $\mathrm{E}_8 \times \mathrm{E}_8$ discussion, we find that there are 131 shift embeddings. The total number of inequivalent models is very similar to the $\mathrm{E}_8 \times \mathrm{E}_8$ case.

6. Discussion

Aiming at a systematic understanding of heterotic $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds, we have investigated the possibilities arising in these constructions. We find that, unlike in the case of prime orbifolds, adding elements of the weight lattice to the shifts, $(V_1, V_2) \to (V_1 + \Delta V_1, V_2 + \Delta V_2)$, changes in general the spectrum. Interestingly, the same spectra are obtained by switching on discrete torsion. Stated differently, one can trade discrete torsion for a change of the gauge embedding by lattice vectors. We have extended our analysis such as to include generalized discrete torsion. We find that a large class of generalized discrete torsion phases can be mimicked by lattice-valued Wilson lines. Interestingly, an analog of a generalized discrete torsion phase appears in certain $\mathbb{Z}_N$ orbifolds which admit at least two discrete Wilson lines of coinciding orders. Another remarkable result is that switching on certain types of generalized discrete torsion is equivalent to changing the 6D compactification lattice. This implies that the field-theoretic interpretation of the model input can be somewhat subtle. We provided various explicit examples, illustrating all main statements.

Our results have important consequences. At the more practical side, we were able to formulate a straightforward method of classifying $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifolds. We have also seen that switching on generalized discrete torsion allows to change the local spectra, i.e. one obtains different twisted states, which correspond to brane fields in the field-theoretic description. We expect this to become important for orbifold model building, where one can use this observation, for instance, for reducing the number of generations without modifying the gauge group.

At a more conceptual level, our findings imply that a given spectrum cannot be ascribed neither to properties of the 6D internal space alone, i.e. whether discrete torsion is switched on or not, nor to the gauge embedding, but only to both. This implies that the same models, leading to the same spectra, can be regarded as resulting from what one might consider as different geometries. Although we cannot claim to have identified the deeper reasons for these relations, we feel that our observations constitute some progress in the task of understanding stringy geometry.

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A. Transformation phases

The aim of this appendix is to clarify the transformation law of physical states. Let us start with the simplest example, a $\mathbb{Z}_N$ orbifold without Wilson lines. Modular invariance requires [3]

$$N\,(V^2 - v^2) \;=\; 0 \quad \mathrm{mod}\ 2\,. \tag{A.1}$$

To see that there are some subtleties, consider a ‘constructing element’ $g = (\theta^k, n_\alpha e_\alpha)$. Zero-modes in the g-twisted sector have to fulfill the masslessness condition (2.10). Focus, for simplicity, on non-oscillator states, for which the masslessness conditions read

$$\frac{1}{2}p_{\mathrm{sh}}^2 - 1 + \delta c = 0\,, \tag{A.2a}$$

$$\frac{1}{2}q_{\mathrm{sh}}^2 - \frac{1}{2} + \delta c = 0\,, \tag{A.2b}$$

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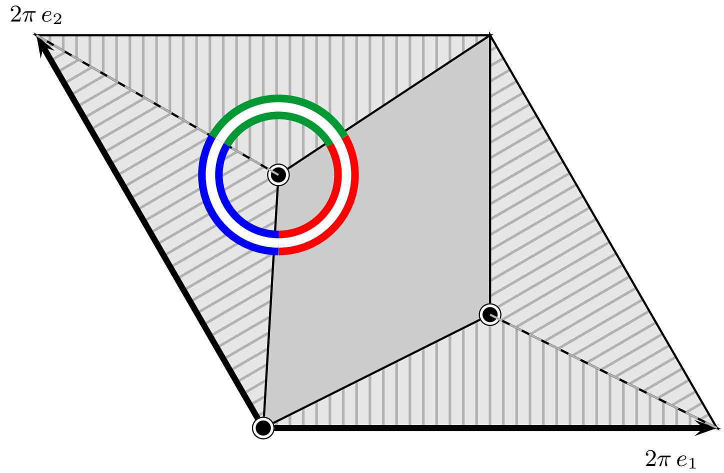

Figure 3: $\mathbb{T}^2_{\mathrm{SU}(3)}/\mathbb{Z}_3$ orbifold. The fundamental domain of the torus can be taken to be the darker area. Transformation of $g = (\theta, e_1 + e_2)$-twisted strings under (the constructing element) g: the (red) right string is mapped to the (green) upper, and the (green) upper to the (blue) left, and the (blue) left back to the (red) upper one.

where $p_{\mathrm{sh}} = (p + V_g)$, $q_{\mathrm{sh}} = (q + v_g)$, $V_g = k\,V + n_\alpha\,A_\alpha$ and $v_g = k\,v$. Using the properties $p^2 =$ even and $q^2 =$ odd one derives

$$p \cdot V_g \;=\; 1 - \delta c - \frac{V_g^2}{2} + \text{integer}\;, \tag{A.3a}$$

$$q \cdot v_g \;=\; -\delta c - \frac{v_g^2}{2} + \text{integer}\;. \tag{A.3b}$$

Now let us study the transformation of a solution of the mass equation $|\psi\rangle$. The shifted momenta p_{sh} and q_{sh} represent the gauge and Lorentz (or SO(8)) quantum numbers. This fixes the transformation phase of the g-twisted string $|\psi\rangle$ to be

$$2\pi\,[p_{\mathrm{sh}} \cdot V_g - q_{\mathrm{sh}} \cdot v_g] \;=\; 2\pi \left[\frac{1}{2}\left(V_g^2 - v_g^2\right) \mod 1\right] \tag{A.4}$$

under the action of g. From (A.1) we infer that this phase does not vanish in general, rather it is of the form $2\pi\,m/N$ with $m \in \mathbb{Z}$. This raises the question of how a state associated to g can be invariant under the action of g. In the literature, the transformation behavior of states associated to twisted strings has often been 'repaired', i.e. an additional transformation phase has been introduced by hand. In what follows, we present a geometric explanation of how such additional phases arise.

Consider a string twisted by the space group element g in the 'upstairs' picture, i.e. on the torus. Twisted strings end at the borders of the fundamental domain of the orbifold. The fundamental domain of the orbifold together with its non-trivial images under g, g^2 etc. comprise the fundamental domain of the torus. Therefore, on the torus a twisted string appears in n copies where n is the order of g, i.e. the minimal positive integer with $g^n = \mathbb{1}$. A 2D illustration is shown in figure 3. That is, a g-twisted state appears as a linear combination of the state $|\psi\rangle$ corresponding to the g-twisted string in the fundamental domain and its images under g^m. This linear combination can involve phase factors with the constraint being that g^n acts as identity. There are n different linear combinations labeled by $m \in \mathbb{Z}$,

$$|\psi^{(m)}\rangle = \frac{1}{\sqrt{n}} \left(|\psi\rangle + e^{-2\pi \mathrm{i}\, m/n} \,|g\,\psi\rangle + \cdots + e^{-2\pi \mathrm{i}\, m\,(n-1)/n} \,|g^{n-1}\,\psi\rangle \right) , \tag{A.5}$$

on the torus. In figure 3 this would correspond to a superposition of the red, green and blue string, weighted by phase factors. It is clear that, under the action of g, such a linear combination picks up a phase $e^{2\pi \mathrm{i}\, m/n}$. In other words, the transformation phase (A.4) is to be amended by $2\pi\, m/n$.

It is straightforward to apply these observations to the above-mentioned problem of g-invariance of states associated to the constructing element g. Clearly, only the linear combination (A.5) with $m = -\frac{n}{2}(V_g^2 - v_g^2)$ is g-invariant. So we conclude that for every solution of the mass equation one finds precisely one g-invariant state. In addition to the phase arising from the gauge and Lorentz quantum numbers (A.4), this state picks up a compensating phase

$$\Phi_{\mathrm{vac}} = \exp\left\{2\pi \mathrm{i} \left[-\frac{1}{2}(V_g \cdot V_g - v_g \cdot v_g)\right]\right\} \tag{A.6}$$

under the action of g.

Before turning to the discussion of the transformation of such states under commuting elements h, let us briefly comment on a technical simplification that is possible in many models. It is also clear that, in $\mathbb{Z}_N$ orbifolds without Wilson lines, one can 'transform' a model M, with a 'weak' shift V, to the model M$'$, with shift $V' = V + \Delta V$ where $\Delta V \in \Lambda$. The solutions of the mass equation coincide in the models M and M$'$. It is, up to some exceptional cases which we will discuss below, always possible to find a ΔV with $\frac{1}{2}[(V+\Delta V)^2 - v^2] \in \mathbb{Z}$ so that (A.4) is automatically fulfilled for any $|\psi\rangle$ (and therefore also for trivial linear combinations). For a large class of constructions one can therefore adopt the following logic. Models with an input fulfilling only the ('weak') modular invariance constraints (A.1) might not be considered as they have an alternative description in terms of a model with input fulfilling the stronger constraints

$$V^2 - v^2 = 0 \mod 2 . \tag{A.7}$$

That is, in order to avoid double-counting, one can restrict to the stronger constraints (A.7) in many cases. Similar statements apply to the case with non-trivial Wilson lines (an example has been given in [13]). However, there is a caveat, namely the above-mentioned exceptional cases in which a 'weak' modular invariant input cannot be transformed to the 'strong' form. The simplest example for such a case is a $\mathbb{Z}_3$ orbifold with $V = 0$. Further examples arise in non-prime $\mathbb{Z}_{N\cdot M}$ orbifolds $(N, M > 1)$ where $N \cdot V \in \Lambda$.

Let us now return to the question of how a (g-invariant) state associated to the constructing element g transforms under the action of a commuting element h. In the following, we denote the corresponding transformation phase of equation (2.13) by $\Phi(g, h)$. Clearly, this phase has to comply with the space group multiplication law, thus

$$\Phi_{\mathrm{vac}}(g,\, g^p\, h^q) = [\Phi_{\mathrm{vac}}(g,\, g)]^p\, [\Phi_{\mathrm{vac}}(g,\, h)]^q . \tag{A.8}$$

Since $\Phi_{\text{vac}}(g,\, g)$ is already fixed by (A.6), this leads to the conclusion

$$\Phi_{\text{vac}}(g,\, h) \;=\; \exp\left\{2\pi\mathrm{i}\left[-\frac{1}{2}\left(V_g \cdot V_h - v_g \cdot v_h\right)\right]\right\} \tag{A.9}$$

for any commuting $h \in S$. Therefore, the full transformation phase of the physical states has to be defined as in equation (2.13). But there are still some constraints which have to be fulfilled for the sake of consistency. To illustrate them, let us consider $g, h \in S$ with $g^n = \mathbb{1} = h^s$. From the definition of the full transformation phase Φ, it is clear that one has to demand

$$\Phi(g,h) \;\stackrel{!}{=}\; \Phi(g^{n+1},h) \;=\; \Phi(g,h)\,\Phi_{\text{vac}}(g,h)^{-n}\,, \tag{A.10}$$

where the second equality is obtained by replacing, according to the usual embedding, $V_g \to (n+1)\,V_g$ and $v_g \to (n+1)\,v_g$ in equation (2.13). Thus, $\Phi_{\text{vac}}(g,h)^n \stackrel{!}{=} 1$. An analogous reasoning starting with $\Phi(g,h^{s+1})$ leads to $\Phi_{\text{vac}}(g,h)^s \stackrel{!}{=} 1$ and thus finally to

$$\Phi_{\text{vac}}(g,h)^{\gcd(n,s)} \;\stackrel{!}{=}\; 1\,. \tag{A.11}$$

Formulating equation (A.11) in terms of the gauge embedding shifts leads to the consistency conditions (2.7) on shifts and Wilson lines.[6]

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